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THE MATHEMATICAL THEORY OF THE TOP

Physics
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MATHEMATICAL THEORY OF THE TOP

LECTURES DELIVERED ON THE OCCASION OF THE SESQUICENTENNIAL CELEBRATION OF PRINCETON UNIVERSITY

BY

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WITH ILLUSTRATIONS

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NOTE

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LECTURE I

In the following lectures it is proposed to consider certain interesting and important questions of dynamics from the standpoint of the theory of functions of the complex variable. I am to develop a new method, which, as I think, renders the discussion of these questions simpler and more attractive. My object in presenting it, however, is more general than that of throwing light on a particular class of problems in dynamics. I wish by an illustration which may fairly be regarded as representative to make evident the advantage which is to be gained by dynamics and astronomical and physical science in general from a more intimate association with the modern pure mathematics, the theory of functions especially.

I venture to hope, therefore, that my lectures may interest engineers, physicists, and astronomers as well as mathematicians. If one may accuse mathematicians as a class of ignoring the mathematical problems of the modern physics and astronomy, one may, with no less justice perhaps, accuse

physicists and astronomers of ignoring departments of the pure mathematics which have reached a high degree of development and are fitted to render valuable service to physics and astronomy. It is the great need of the present in mathematical science that the pure science and those departments of physical science in which it finds its most important applications should again be brought into the intimate association which proved so fruitful in the work of Lagrange and Gauss.

I shall confine my discussion mainly to the problem presented in the motion of a top — meaning for the present by "top" a rigid body rotating about an axis, when a single point in this axis, not the centre of gravity, is fixed in position.

In the present lecture I shall present some preliminary considerations of a purely geometrical character. But it is necessary first of all to obtain an analytical representation of the rotation of a rigid body about a fixed point, and I shall begin with a statement of the methods ordinarily used.

We introduce two systems of rectangular axes both having their origin at the fixed point: the one system, x, y, z, fixed in space; the other, X, Y, Z, fixed in the rotating body. Then the ordinary equations of transformation from the one

system to the other, which may be exhibited in the scheme:

$$(1) \qquad \begin{array}{c|ccc} & X & Y & Z \\ \hline x & a & b & c \\ y & a' & b' & c' \\ z & a'' & b'' & c'' \end{array}$$

give at once, when the nine direction cosines, $a, b, c, a', \cdots$ are known functions of the time t, the representation of the motion of the movable system X, Y, Z, with respect to the fixed system x, y, z.

As is well known, these cosines are not independent; they are rather functions of but three independent quantities or parameters. It is customary to employ one or other of the following sets of parameters, both of which were introduced by Euler.

The first set of parameters, which is non-symmetrical, consists of the angle ϑ which the Z-axis makes with the z-axis, and the angles ϕ and ψ, which the line of intersection of the xy- and XY-planes makes with the X-axis and the x-axis respectively. Because of the frequent use made of these parameters in astronomy, I shall call them the "astronomical parameters." When the cosines $a, b, c, \cdots$ have been expressed in terms

of them, the orthogonal substitution (1) becomes:

$$
(2)\qquad
\begin{array}{c|ccc}
 & X & Y & Z \\
\hline
x & \cos\phi\cos\psi - \cos\vartheta\sin\phi\sin\psi, & -\sin\phi\cos\psi - \cos\vartheta\cos\phi\sin\psi, & \sin\vartheta\sin\psi \\
y & \cos\phi\sin\psi + \cos\vartheta\sin\phi\cos\psi, & -\sin\phi\sin\psi + \cos\vartheta\cos\phi\cos\psi, & -\sin\vartheta\cos\psi \\
z & \sin\vartheta\sin\phi\qquad, & \sin\vartheta\cos\phi\qquad, & \cos\vartheta
\end{array}
$$

The second set of parameters may be defined as follows. Every displacement of our body is equivalent to a simple rotation about a fixed axis. Let ω be the angle of rotation, and a, b, c the angles which the axis makes with OX, OY, OZ; and set

$$A = \cos a \sin\frac{\omega}{2},\ B = \cos b \sin\frac{\omega}{2},\ C = \cos c \sin\frac{\omega}{2},\ D = \cos\frac{\omega}{2}.$$

The quantities A, B, C, D (of which but three are independent, since, as will be seen at once, $A^2 + B^2 + C^2 + D^2 = 1$) are the parameters under consideration. In terms of them our orthogonal substitution (1) is

$$
(3)\qquad
\begin{array}{c|lll}
 & X & Y & Z \\
\hline
x & D^2+A^2-B^2-C^2, & 2(AB-CD), & 2(AC+BD) \\
y & 2(AB+CD), & D^2+B^2-C^2-A^2, & 2(BC-AD) \\
z & 2(AC-BD), & 2(BC+AD) & D^2+C^2-A^2-B^2
\end{array}
$$

or, if use be not made of the relation

$$A^2 + B^2 + C^2 + D^2 = 1,$$

a substitution with these coefficients each divided by $A^2 + B^2 + C^2 + D^2$. I shall call these the

"quaternion parameters," inasmuch as the quaternionists make frequent use of them. The quaternion corresponding to our rotation is

$$q = D + iA + jB + kC.$$

These parameters are very symmetrical, and for that reason very attractive. Nevertheless, they do not prove to be the most advantageous system for our present purpose. Our problem is not a symmetrical problem. In it one of the axes, Oz, in the direction of gravity, plays an exceptional rôle; the motion of the top is not isotropic.

Instead of either of these commonly used systems of parameters, I propose to introduce another, which so far as I know has not yet been employed in dynamics.

Let x, y, z be the coordinates of a point on a sphere fixed in space which has the radius r and the centre O, and X, Y, Z the coordinates of a point on a sphere congruent with the first but fixed in the rotating body. As the body rotates, the second sphere slides about on the first, but remains always in congruence with it.

It is characteristic of every point on the first sphere that the relation

$$\frac{x + iy}{r - z} = \frac{r + z}{x - iy}$$

holds good between its coordinates.

If we represent the values of these equal ratios by ζ, obviously ζ is a parameter for the points of the sphere, which completely determines one of these points for every value that it may take. Thus the upper extremity of the z-axis is characterized by the value ∞ of ζ, the lower extremity by the value 0; to real values of ζ correspond the points on the great circle of the sphere in the plane $y=0$, and to pure imaginary values the points of the great circle in the plane $x=0$.

For the points of the second sphere, in like manner, there is a parameter Z connected with the coordinates X, Y, Z by the equations,

$$\frac{X+iY}{r-Z}=\frac{r+Z}{X-iY}=\mathrm{Z},$$

which defines these points as ζ defined the points of the fixed sphere.

If now ζ and Z be parameters of corresponding points on the two spheres, what is the relation between these parameters when the second sphere is subjected to a rotation? *It is a simple linear relation of the form*

$$\zeta=\frac{\alpha\mathrm{Z}+\beta}{\gamma\mathrm{Z}+\delta},$$

in which α, β, γ, δ *are themselves in general complex quantities, but so related that* α *is the conjugate im-*

aginary to δ, *and* β *to* $-\gamma$; *or, adopting the ordinary notation,* $\alpha = \bar{\delta}$ *and* $\beta = -\bar{\gamma}$.

It is obvious, *a priori*, that the relation must be linear, and a very simple reckoning such as I have given in my treatise on the Icosahedron (p. 32) establishes the special relations among the coefficients. There are but four real quantities involved in α, β, γ, δ, only the ratios of which need be considered independent, since these ratios alone appear in the expression for ζ; unless, as is generally more convenient, we introduce the further relation $\alpha\delta - \beta\gamma = 1$.

It is these quantities α, β, γ, δ connected by the relation $\alpha\delta - \beta\gamma = 1$, which together with ζ we propose to use as our parameters in the discussion of the problem now under consideration. They were introduced into mathematics by Riemann forty years ago, and have proved to be peculiarly useful in different geometrical problems intimately connected with the theory of functions, especially in the theory of minimal surfaces and the theory of the regular solids. *We hope to show that they may be employed to quite as great advantage in the study of all problems connected with the motion of a rigid body about a fixed point.*

Corresponding to the orthogonal substitution (1), we have in terms of our new parameters the substitution

$$
(4)\qquad
\begin{array}{r|ccc}
 & X+iY & -Z & -X+iY \\
\hline
x+iy & \alpha^2 & 2\,\alpha\beta & \beta^2 \\
-z & \alpha\gamma & \alpha\delta+\beta\gamma & \beta\delta \\
-x+iy & \gamma^2 & 2\,\gamma\delta & \delta^2
\end{array}
$$

as may be demonstrated without serious reckoning as follows. And I may remark incidentally that it seems to me better wherever possible to effect a mathematical demonstration by general considerations which bring to light its inner meaning rather than by a detailed reckoning, every step in which the mind may be forced to accept as incontrovertible, and yet have no understanding of its real significance.

Consider the sphere of radius 0,

$$x^2+y^2+z^2=0.$$

It is an imaginary cone whose generating lines join the origin to the so-called "imaginary circle at infinity," the circle in which all spheres intersect at infinity. For this sphere,

$$\zeta=\frac{x+iy}{-z}=\frac{z}{x-iy},$$

or
$$x+iy:-z:x-iy=\zeta^2:\zeta:-1.$$

Here to each value of the parameter ζ there corresponds a single (imaginary) generating line of the cone, and *vice versa*. In other words, there is a relation of one-to-one correspondence between

the (imaginary) generating lines of the cone and the values of ζ, or the cone is unicursal.

There is, of course, the same relation between the generating lines of the congruent cone

$$X^2 + Y^2 + Z^2 = 0,$$

which is fixed in the moving body, and the parameter

$$\mathrm{Z} = \frac{X + iY}{-Z} = \frac{Z}{X - iY}.$$

When the body rotates, this cone is simply carried over into itself, so that the generating lines in their new position are in one-to-one correspondence with the same generating lines in their original position. Between the parameters Z and ζ, which correspond to the generating lines in these two positions, there is, therefore, also a relation of one-to-one correspondence, or the two are connected linearly, *i.e.* by a relation of the form:

$$\zeta = \frac{\alpha \mathrm{Z} + \beta}{\gamma \mathrm{Z} + \delta},$$

where, as above, we suppose

$$\alpha\delta - \beta\gamma = 1.$$

If now we avail ourselves of the advantages to be had from the use of homogeneous equations and substitutions by replacing

$$\zeta \text{ by } \frac{\zeta_1}{\zeta_2}, \text{ and } \mathrm{Z} \text{ by } \frac{\mathrm{Z}_1}{\mathrm{Z}_2},$$

this single equation may be replaced by the two homogeneous equations:

$$\zeta_1 = \alpha \mathrm{Z}_1 + \beta \mathrm{Z}_2,$$
$$\zeta_2 = \gamma \mathrm{Z}_1 + \delta \mathrm{Z}_2,$$

and the equations connecting x, y, z, and ζ, and X, Y, Z, and Z become,

$$x + iy : -z : -x + iy = \zeta_1^2 : \zeta_1\zeta_2 : \zeta_2^2,$$
$$X + iY : -Z : -X + iY = \mathrm{Z}_1^2 : \mathrm{Z}_1\mathrm{Z}_2 : \mathrm{Z}_2^2.$$

From these equations it follows that

$$x + iy = \alpha^2(X + iY) + 2\,\alpha\beta(-Z) + \beta^2(-X + iY)$$
$$-z = \alpha\gamma(X + iY) + (\alpha\delta + \beta\gamma)(-Z) + \beta\delta(-X + iY)$$
$$-x + iy = \gamma^2(X + iY) + 2\,\gamma\delta(-Z) + \delta^2(-X + iY).$$

For it is immediately obvious that $x + iy$ is proportional to ζ_1^2, therefore to

$$\alpha^2\mathrm{Z}_1^2 + 2\,\alpha\beta\mathrm{Z}_1\mathrm{Z}_2 + \beta^2\mathrm{Z}_2^2,$$

and therefore finally to

$$\alpha^2(X + iY) + 2\,\alpha\beta\,(-Z) + \beta^2(-X + iY);$$

and in like manner, that $-z$ and $-x + iy$ are proportional to

$$\alpha\gamma\,(X + iY) + (\alpha\delta + \beta\gamma)(-Z) + \beta\delta(-X + iY),$$

and

$$\gamma^2(X + iY) + 2\,\gamma\delta\,(-Z) + \delta^2(-X + iY)$$

respectively. And that $x + iy$, $-z$, $-x + iy$ are severally *equal* to these expressions and not merely proportional to them, follows from the fact that the determinant of the orthogonal substitution connecting x, y, z with X, Y, Z must equal 1.

The demonstration, to be sure, applies directly to the points of the imaginary cone only. But it is known in advance that the transformation which we are considering is a linear one for *all* points of space. Its coefficients are the same for all points, and we have merely availed ourselves of the fact that the imaginary cone remains unchanged by the transformation to determine them. The same result might have been reached, though less simply, by using the general formula $\zeta = \dfrac{x + iy}{r - z}$. The equations (4), therefore, are those which connect the coordinates of the initial and final positions of any point rigidly attached to the rotating body.

The relations between our new parameters, α, β, γ, δ, and the astronomical parameters, ϑ, ϕ, ψ, on the one hand, and the quaternion parameters A, B, C, D, on the other, are of immediate interest and of importance in the subsequent discussion. They are to be had very simply by a comparison of the coefficients in the three schemes (2), (3), (4), and, after reduction, prove to be:

$$(5)\quad \begin{cases} \alpha = \cos\frac{\vartheta}{2}\cdot e^{\frac{i(\phi+\psi)}{2}}, & \beta = i\sin\frac{\vartheta}{2}\cdot e^{\frac{i(-\phi+\psi)}{2}}, \\ \gamma = i\sin\frac{\vartheta}{2}\cdot e^{\frac{i(\phi-\psi)}{2}}, & \delta = \cos\frac{\vartheta}{2}\cdot e^{\frac{-i(\phi+\psi)}{2}}, \end{cases}$$

and $$\begin{cases} \alpha = D + iC, & \beta = -B + iA, \\ \gamma = B + iA, & \delta = D - iC. \end{cases}$$

Our new parameters are thus imaginary combinations of the real parameters in ordinary use. Mathematical physics affords many examples of the advantage to be gained by employing such imaginary combinations of real quantities. It is only necessary to cite the use made of them in optics by Cauchy.

I may remark that Darboux in his *Leçons sur la théorie générale des surfaces*, Livre I., treats the subject of rotation in a manner which is very similar to that which we have followed. But with him the ζ itself is considered directly as a function of the time and not the separate coefficients $\alpha, \beta, \gamma, \delta$. His method thus lacks the simplicity which is possible when these are made the primary functions.

We now turn to a brief consideration of the meaning of the substitution

$$\zeta = \frac{\alpha Z + \beta}{\gamma Z + \delta},$$

when $\alpha, \beta, \gamma, \delta$ are still regarded as functions of the time, but are general complex quantities, not connected by the special relations $\alpha = \bar{\delta}$, $\beta = -\bar{\gamma}$.

We shall consider t also as capable of complex values, not for the sake of studying the behavior of a fictitious, imaginary time, but because it is only by taking this step that it becomes possible to bring about the intimate association of kinetics and the theory of functions of a complex variable at which we are aiming.

What is the meaning of the above formula? It is still a real transformation of the sphere on which we have defined ζ into itself, a linear transformation in which the coefficients are all real.

If the radius of the sphere be 1, as we shall assume throughout the discussion of this general transformation, or its equation when written homogeneously, be:

$$x^2 + y^2 + z^2 - t^2 = 0,$$

the equations connecting x, y, z, t and X, Y, Z, T are those indicated in the following scheme:

(6)

	$X+iY$	$X-iY$	$T+Z$	$T-Z$
$x+iy$	$\alpha\bar{\delta}$	$\beta\bar{\gamma}$	$\alpha\bar{\gamma}$	$\beta\bar{\delta}$
$x-iy$	$\gamma\bar{\beta}$	$\delta\bar{\alpha}$	$\gamma\bar{\alpha}$	$\delta\bar{\beta}$
$t+z$	$\alpha\bar{\beta}$	$\beta\bar{\alpha}$	$\alpha\bar{\alpha}$	$\beta\bar{\beta}$
$t-z$	$\gamma\bar{\delta}$	$\delta\bar{\gamma}$	$\gamma\bar{\gamma}$	$\delta\bar{\delta}$

and when these equations are solved for x, y, z, t, in terms of X, Y, Z, T, it will be found that the coefficients are real, as has been already stated.

This scheme may be derived in a manner analogous to that followed in deriving the scheme (4).

The equation of the sphere

$$x^2 + y^2 + z^2 - t^2 = 0,$$

or $$(x + iy)(x - iy) + (z + t)(z - t) = 0,$$

may, as is readily verified, be written in the form,

$$x + iy : x - iy : t + z : t - z = \zeta_1\zeta_2' : \zeta_2\zeta_1' : \zeta_1\zeta_1' : \zeta_2\zeta_2',$$

where $\dfrac{\zeta_1}{\zeta_2} = \zeta$, and ζ_1', ζ_2' are, for real values of x, y, z, t, the conjugate imaginaries to ζ_1, ζ_2 respectively.

As above, $$\zeta = \frac{x + iy}{t - z} = \frac{t + z}{x - iy}.$$

If then Z_1, Z_2, Z_1', Z_2' be quantities similarly defined with respect to the movable sphere

$$X^2 + Y^2 + Z^2 - T^2 = 0,$$

we have corresponding to the transformation

$$\zeta = \frac{\alpha Z + \beta}{\gamma Z + \delta},$$

the two pairs of equations:

$$\zeta_1 = \alpha Z_1 + \beta Z_2, \qquad \zeta_1' = \bar{\alpha} Z_1' + \bar{\beta} Z_2',$$
$$\zeta_2 = \gamma Z_1 + \delta Z_2, \qquad \zeta_2' = \bar{\gamma} Z_1' + \bar{\delta} Z_2',$$

if the transformation is to be real.

And from this series of equations it follows by the reasoning used on page 10 that $x+iy$ is equal to

$$\alpha\bar{\delta}(X+iY)+\beta\bar{\gamma}(X+iY)+\alpha\bar{\gamma}(T+Z)+\beta\bar{\delta}(T-Z),$$

and $x-iy$, $t+z$, $t-z$ to the corresponding expressions indicated in scheme (6).

The scheme (6) at once reduces to the scheme (4) when the special supposition is made that $\alpha=\bar{\delta}$ and $\beta=-\bar{\gamma}$. And since this is the sufficient and necessary condition that (6) reduce to (4), we have here an independent demonstration that these relations hold good among the parameters α, β, γ, δ when the motion is a rotation about a fixed point.

The general transformation (6) represents the totality of those projective transformations or collineations of space for which each system of generating lines of the sphere, $x^2+y^2+z^2-t^2=0$, is transformed into itself, and among which all rotations of the sphere are obviously included as special cases. This is the geometrical meaning of the equation

$$\zeta=\frac{\alpha Z+\beta}{\gamma Z+\delta}$$

for unrestricted values of α, β, γ, δ.

But the transformation admits also of a very interesting kinematical interpretation which I shall consider at length in my third lecture. With

respect to it our sphere of radius 1 plays the rôle of the fundamental surface or "absolute" in the Cayleyan or hyperbolic non-Euclidian geometry. For any free motion in such a space the absolute remains fixed in position as in ordinary space the imaginary circle at infinity $x^2 + y^2 + z^2 = 0$, $t = 0$, does, which is its absolute.

The transformation therefore represents a real free motion in non-Euclidean space, and the six independent real parameters involved in the ratios $\alpha : \beta : \gamma : \delta$ correspond to the ∞^6 such possible motions. Interpreted in Euclidean space, the transformation represents a motion of the body combined with a strain.

I close the present lecture with two remarks.

First, *there is nothing essentially new in the considerations with which we have been occupied thus far.* I have merely attempted to throw a method already well known into the most convenient form for application to mechanics.

Second, *the non-Euclidean geometry has no metaphysical significance here or in the subsequent discussion.* It is used solely because it is a convenient method of grouping in geometric form relations which must otherwise remain hidden in formulas.

LECTURE II

I NOW proceed at once to the discussion of the Lagrange equations of motion for our top, only pausing to remark once more that this problem of the top is for us typical of all dynamical questions which are related to a sphere. To this category belong also the problem of the spherical pendulum (which in fact is a special case of the problem of the top), the problem of the catenary on the sphere, and all problems of the motion of a rigid body about a fixed point. The simplest problem of the type is that of the motion of a rigid body about its centre of gravity, the Poinsot motion, as we shall name it after Poinsot who treated it very elegantly.

We shall first state the equations in terms of the astronomical parameters; and to give the expressions as simple a form as possible, I shall suppose the principal moments of inertia of the top about the fixed point of support each equal to 1. One may call such a top a spherical top, as its momental ellipsoid is a sphere. I wish it understood, however, that this restriction is not essential to the

application of our method, but is rather made solely for the sake of rendering its presentation more easy.

On this assumption, we have for the kinetic energy, T, of the motion the expression

$$T = \tfrac{1}{2}(\phi'^2 + \psi'^2 + 2\,\phi'\psi' \cos\vartheta + \vartheta'^2),$$

where ϑ', ϕ', ψ' are the derivatives of ϑ, ϕ, ψ with respect to t; and for the potential energy, V, the expression

$$V = P\cos\vartheta,$$

where P represents the static moment of the top with respect to O.

The Lagrange equations are:

$$\frac{d\dfrac{\delta T}{\delta\phi'}}{dt} = 0, \quad \frac{d\dfrac{\delta T}{\delta\psi'}}{dt} = 0, \quad \frac{d\dfrac{\delta T}{\delta\vartheta'}}{dt} = \frac{\delta(T-V)}{\delta\vartheta}.$$

The first two equations are especially simple in having their right members equal to zero, and we are therefore able to derive immediately the two algebraic first integrals

$$\phi' + \psi'\cos\vartheta = n,$$
$$\psi' + \phi'\cos\vartheta = l.$$

The quantities n and l are constants of integration, to be determined from the initial conditions of the motion. In the following discussion we shall suppose them positive.

In addition to these integrals, we have the equation of energy

$$T + V = h,$$

where h also is a constant determined, like l and n, by the special conditions of the problem.

Solving the first two equations for ϕ' and ψ', and substituting the results in the third, and setting $\cos\vartheta = u$, and

$$U = 2\,Pu^3 - 2\,hu^2 + 2\,(ln - P)\,u + 2\,h - l^2 - n^2,$$

we obtain finally for t, ϕ, and ψ, expressed as functions of u, the formulas

$$t = \int \frac{du}{\sqrt{U}}, \quad \phi = \int \frac{n - lu}{1 - u^2}\,\frac{du}{\sqrt{U}}, \quad \psi = \int \frac{l - nu}{1 - u^2}\,\frac{du}{\sqrt{U}}.$$

The problem of the motion of the top is thus reduced to three simple integrations or quadratures, as indeed was demonstrated by Lagrange himself. These integrals are elliptic integrals, U being a polynomial of the third degree in u, the first an elliptic integral of the "first kind" (which is characterized by being finite for all values of the independent variable), the remaining two elliptic integrals of a more complex character.

It is often said that dynamics reached its ultimate form in the hands of Lagrange, and the cry "return to Lagrange" is frequently raised by those who set little store by the value for physical

science of recent developments in the pure mathematics. But this is by no means just. Lagrange reduced our problem to quadratures, but Jacobi made a great stride beyond him, as we mathematicians think, by introducing the elliptic functions, which enabled him to assign to t the rôle of independent variable and to discuss the remaining variables u, ϕ, ψ directly as functions of the time. *An advantage was thus gained not only for the understanding of the essential relations of the variables to one another, but for simplicity of computation also.* The coefficients a, b, c, a', $\cdots$, are *uniform* (or one valued) functions of t, and one of the most useful properties a function can possess, if its values must be computed, is that it be uniform. This work of Jacobi is not as well known as it should be, having first appeared posthumously, in the second volume of his collected works, published by the Berlin Academy in 1882. I may add that his pupils, Lottner and Somoff, developed the same method in papers published in 1855 independently. It is shown in these papers that the nine cosines a, b, c, $\cdots$, may be expressed in terms of theta functions.*

* As is well known, Jacobi gave analogous formulas for the nine cosines of the Poinsot motion in 1849. Closely related to this representation of the cosines is the interesting theorem to which we shall return later on, that the motion of our top may be reproduced by compounding two Poinsot motions.

But while the a, b, c, $\cdots$, considered as functions of t, are much simpler than the integrals of Lagrange, they are at the same time much more complicated than our parameters α, β, γ, δ. These parameters prove to be the simplest possible elliptic functions of t; so that by introducing them we carry to its completion the work begun by Jacobi, of reducing our problem to its simplest elements.

For the proper understanding of this treatment of the motion of the top, some knowledge of the nature of elliptic functions is obviously necessary; and I know of no readier means of gaining this than *Riemann's method of conformal representation*, — of which, moreover, we shall have other important applications to make later on.

In accordance with this method, we construct the "Riemann surface" of the function $\sqrt{U}$ on the plane of the complex variable u, in the following manner: The polynomial U vanishes for three values of u, all of which may readily be shown to be real, and becomes infinite when $u = \infty$. Two of these roots, e_1, e_2, lie between -1 and $+1$; and the third, e_3, between $+1$ and ∞. Therefore, $\sqrt{U}$ is a two-valued function of u everywhere in the u-plane, except at the four points of the real axis, e_1, e_2, e_3, e_∞. To obtain a surface, therefore, between whose points and the values of $\sqrt{U}$ there shall be a one-to-one correspondence, we lay over the u-plane

two sheets, which are everywhere distinct except at the points e_1, e_2, e_3, e_∞, in which they coalesce, and associate with the points in the two sheets which lie immediately over any point u in the u-plane, the two corresponding values of $\sqrt{U}$, one with each.

It will be found that if the point u describe any simple circuit in the u-plane, which encloses one and but one of the points e_1, e_2, e_3, e_∞, returning finally to its initial position, $\sqrt{U}$ will pass from the one to the other of the two values which correspond to the initial value of u; the point corresponding to u in the Riemann surface of $\sqrt{U}$, must, therefore, move from a position in the one sheet to a position immediately under (or over) this in the other. But this is possible only if we suppose the two sheets to cross along some line running out from each of the points, e_1, e_2, e_3, e_∞, — not to intersect, but to cross, as non-intersecting lines in space may be said to cross. Inasmuch as this is the simplest hypothesis possible, we shall take as these lines of crossing, in the present case, the segments, e_1e_2, e_3e_∞, of the real axis; and have, as a rough representation of the Riemann surface of $\sqrt{U}$, the following figure:

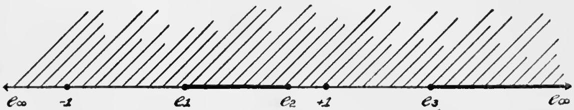

Fig. 1.

where we have shaded the positive half-sheets of the surface and have marked the segments, e_1e_2, e_3e_∞ by heavy lines.

The points, e_1, e_2, e_3, e_∞, are called the "branch points" and the segments, e_1e_2, e_3e_∞, the "branch lines" of the surface.

To construct in the t-plane the figure which is the conformal representation of this Riemann surface, we conceive of this surface as cut into four half-sheets, by an incision made all along the real axis, and seek first the conformal representation of the upper half-sheet. To obtain this, we cause the point u to move, in the positive sense, along the real axis, from e_1 through $e_2e_3e_\infty$, back (from the left) to e_1 and study the corresponding changes of value of t by means of the integral, $t=\int\frac{du}{\sqrt{U}}$, by which it is defined.

We thus find that as the point u traces out the real axis in its plane, the corresponding t traces out a rectangle in its plane, which we may represent by the figure:

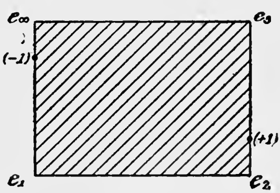

FIG. 2.

to the angular points of which we have attached the values of u to which they correspond, and which we have shaded, since the sense in which its perimeter was traced shows that it is its interior which corresponds to the shaded half-plane of the preceding figure.*

As long as the integral which defines t is left an indefinite integral, this rectangle remains free to occupy any position in the t-plane, — only the directions of its sides, parallel respectively to the real and imaginary axes, and their lengths, — call them ω_1 and ω_2, — are completely determined. But when the integral is made definite, by making e_∞ the lower limit of integration, the angular point, e_∞, coincides with the origin in the t-plane, and the rectangle takes a definite position in the plane.

From the image which we have thus obtained of the one half-sheet, the images of the three remaining half-sheets are to be had at once by the process of "symmetrical reproduction"; which yields for the Riemann surface, when cut in the manner indicated, the complete image:

* The figure is a rectangle since $\sqrt{U}$ is real from $u = e_1$ to $u = e_2$, and from $u = e_3$ to $u = e_\infty$, and a pure imaginary from $u = e_\infty$ to $u = e_1$, and from $u = e_2$ to $u = e_3$. At e_i, $t -$ const. vanishes as $(u - e_i)^{\frac{1}{2}}$.

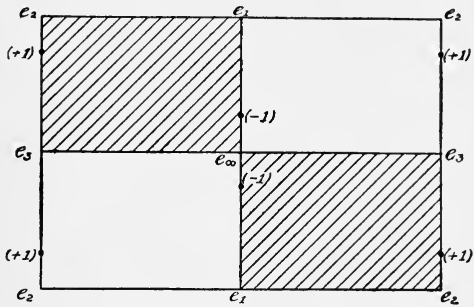

Fig. 3.

The symmetry of the figure with respect to the sides, $e_\infty e_1$, and $e_\infty e_3$, of the original rectangle, will be at once noticed. Each of the four smaller rectangles is the image of one half-sheet; the shaded, of the positive half-sheets; the non-shaded, of the negative.

But we have not yet obtained the complete geometrical representation of u, regarded as a function of t. The Riemann surface of two sheets, which we have thus far been considering, possesses a distinct point for every value of $\sqrt{U}$ regarded as a function of u, but not for t when so regarded. The integral t is affected by an additive constant if u be made to trace in the Riemann surface a closed path which surrounds $e_1 e_2$, or one which surrounds $e_2 e_3$, so that the Riemann surface of t is one possessing the same

branch points as the Riemann surface of $\sqrt{U}$, but having an infinite number of sheets, into any one of which it is possible to move the tracing point, u, if no such cut be made in the surface as that made above along the real axis.

It is a great advantage of the Riemann method that the complete image in the t-plane of this uncut Riemann surface of an infinite number of sheets may be had from the image already obtained for the cut surface, by simply affixing a rectangle, congruent with this image, to each of its sides, repeating the process for the new rectangles, and so on indefinitely, until the entire t-plane is covered by congruent rectangles, any one of which may be brought into coincidence with any other by two translations, one in the direction of the real, the other in the direction of the imaginary, axis.

From the result of this construction, there at once follows a conclusion of the very first importance. The image of the complete Riemann surface of t entirely covers the t-plane, *but without the overlapping of any of its parts.* It follows immediately, therefore, that to each point in the t-plane there corresponds but a single point in the Riemann surface, or that u, and $\sqrt{U}$ as well, is a *uniform* function of t.

The equation connecting t and u is:

$$t = \int_{\infty}^{u} \frac{du}{\sqrt{U}}.$$

And the conclusion which we have reached is, that the functional relation of u with respect to t, defined by this equation, is vastly more simple than that of t with respect to u; to each value of u there corresponded an infinite number of values of t, while to each value of t there corresponds but one value of u. As thus defined, u is called an *elliptic function* of t.

Let ω_1 be the length of the side $e_\infty e_3$ of the small rectangle, which was the image of a half-sheet of the Riemann surface, and ω_2 the length of the side $e_\infty e_1$; then, obviously, if we set $u = \phi(t)$, and t_0 be any point of the complete rectangle (Fig. 3), since $t_0 + m_1 2\omega_1 + m_2 2i\omega_2$ is for any integral values of m_1, m_2, the corresponding point of another of the rectangles, $\phi(t_0 + 2m_1\omega_1 + 2m_2 i\omega_2) = \phi(t_0)$; or the elliptic function, u, is *doubly periodic*, with the periods $2\omega_1$, $2i\omega_2$.

Let us next consider the nature of ϕ and ψ when regarded as functions of t. The integrals by which they are expressed in terms of u are elliptic integrals of greater complexity than is the integral for t. There are on the Riemann surface of $\sqrt{U}$ four points, at which each of these integrals be-

comes logarithmically infinite; namely, the points -1, $+1$, in the upper sheet, and the same points in the lower sheet. Elliptic integrals possessing such points of logarithmic discontinuity are called "elliptic integrals of the third kind," and it is possible to express any such integral in terms of integrals of the first kind and "normal" integrals of the third kind, such, namely, as possess but two points of logarithmic discontinuity with the residues $+1$ and -1 respectively.

But if instead of making this reduction of the integrals directly, we introduce those combinations of ϕ and ψ which constitute our parameters α, β, γ, δ, *a remarkable simplification at once ensues such as renders any further reduction unnecessary.* Surely a preestablished harmony exists between the problem before us, and our parameters α, β, γ, δ.

Since $$\alpha = \cos\frac{\vartheta}{2}\cdot e^{\frac{i(\phi+\psi)}{2}},$$

and $$\cos\frac{\vartheta}{2} = \sqrt{\frac{u+1}{2}},$$

we have immediately

$$\log\alpha = \tfrac{1}{2}\log\frac{1+u}{2} + \frac{i(\phi+\psi)}{2}$$

$$= \int \frac{\sqrt{U} + i(l+n)}{2(u+1)}\cdot\frac{du}{\sqrt{U}} - \tfrac{1}{2}\log 2,$$

when for ϕ and ψ their values are substituted.

And in like manner,

$$\log \beta = \int \frac{\sqrt{U} - i(l-n)}{2(u-1)} \cdot \frac{du}{\sqrt{U}} - \tfrac{1}{2}\log 2,$$

$$\log \gamma = \int \frac{\sqrt{U} + i(l-n)}{2(u-1)} \cdot \frac{du}{\sqrt{U}} - \tfrac{1}{2}\log 2,$$

$$\log \delta = \int \frac{\sqrt{U} - i(l+n)}{2(u+1)} \cdot \frac{du}{\sqrt{U}} - \tfrac{1}{2}\log 2,$$

and *these are all normal integrals of the third kind, each with but two points of logarithmic discontinuity* which are distributed in the rectangle of the periods as indicated in the accompanying figure, if we sup-

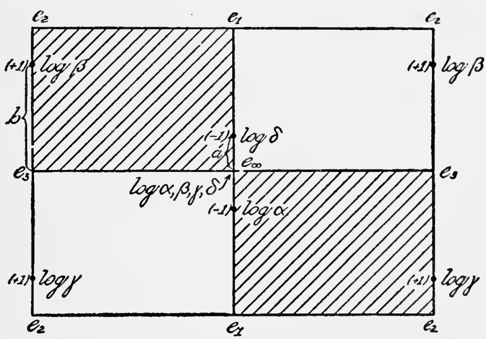

FIG. 4.

pose as we shall find it convenient to do later on that l is less than n.

For $U = -(l+n)^2$ when $u = -1$, and $U = -(l-n)^2$ when $u = +1$. If, therefore, of the two

values of $\sqrt{U}$, which correspond to $u=-1$, we take $i(l+n)$, the factor $\sqrt{U}+i(l+n)$ in the numerator of the expression for $\log\alpha$ will be canceled by the factor $2\sqrt{U}$ in the denominator, while if we take $-i(l+n)$, the numerator vanishes; so that the point -1 in *one* of the sheets of the Riemann surface of $\sqrt{U}$ is the only finite point of discontinuity of the integral $\log\alpha$. It is, moreover, a logarithmic discontinuity with the residue 1, since $\log\alpha$ there becomes ∞ as $\log(u+1)$. On the other hand, for $u=\infty$, *i.e.* at e_∞, $\log\alpha$ becomes infinite as $\frac{1}{2}\log u$. This again is a logarithmic discontinuity, with the residue -1, since e_∞ is at infinity and a branch point. And like considerations apply to the remaining integrals.

By the introduction of the parameters α, β, γ, δ, therefore, the four logarithmic discontinuities of the integrals ϕ, ψ, are assigned one to each of the four normal integrals $\log\alpha$, $\log\beta$, $\log\gamma$, $\log\delta$—normal integrals whose remaining points of discontinuity, corresponding to e_∞, coincide at the origin.

While $\log\alpha$, $\log\beta$, $\log\gamma$, $\log\delta$, as now defined are much simpler functions of u, and therefore of t, than are ϕ and ψ, their exponentials α, β, γ, δ are simpler still. These are uniform functions of t having each one null-point and one ∞-point in every parallelogram of periods. Such functions may always be expressed, apart from an exponential factor, by the quotient of two ϑ- or two σ-functions of the sim-

plest kind — functions which possess one null-point in each parallelogram of periods, but no ∞-point.

Of the ϑ-functions we shall only pause to remark that Jacobi introduced them into analysis as being the simplest elements out of which the elliptic functions could be constructed. He obtained for them expressions in the form of infinite products and infinite series. They are affected by an exponential factor when the argument is increased by a period, but remain otherwise unchanged. The ϑ-functions of the simplest class, with which alone we are concerned, vanish when the argument takes the value zero or a congruent value.

The σ-function of Weierstrass is a more elegant function of the same character.

Inasmuch, therefore, as α, β, γ, δ, are functions of t, which vanish for $t = -ia$, $\omega_1 + ib$, $\omega_1 - ib$, $+ia$ respectively (the values of t corresponding to the points $u = \pm 1$ in the above figure) and which all become infinite for $t = 0$, we have for them the following expressions:

$$\alpha = k_1 e^{\lambda_1 t} \frac{\sigma(t + ia)}{\sigma(t)}, \qquad \beta = k_2 e^{\lambda_2 t} \frac{\sigma(t - \omega_1 - ib)}{\sigma(t)},$$

$$\gamma = k_3 e^{\lambda_3 t} \frac{\sigma(t - \omega_1 + ib)}{\sigma(t)}, \qquad \delta = k_4 e^{\lambda_4 t} \frac{\sigma(t - ia)}{\sigma(t)},$$

where k_i, λ_i are constants to be determined from the initial conditions of the motion. Their values

depend on those of the "transcendental" constants ω_1, ω_2, a, b, as the values of these in turn depend on those of the "algebraic" constants, P, h, l, n.

We shall call functions such as α, β, γ, δ, which miss being doubly periodic by an exponential factor only, "multiplicative elliptic functions." All elliptic functions are expressible as quotients of ϑ- or σ-functions, and evidently of such quotients the simplest possible are those which have a single ϑ or σ of the simplest kind in both numerator and denominator. We may therefore state the result of our discussion in these terms: *We have shown that our parameters* α, β, γ, δ *are multiplicative elliptic functions of the simplest kind, so that by introducing them we have resolved the problem of the top into its simplest elements.*

From these expressions for α, β, γ, δ one may obtain expressions for the nine direction cosines a, b, $c \cdots$ in the form of quotients of σ- or ϑ-functions — such as Jacobi got for them — with the least possible reckoning.*

* Hess has remarked, in his paper on the gyroscope (*Math. Ann.* xxix., 1887) that the *quaternion* expressions for the nine direction cosines are very simple, and our parameters are but linear combinations of the quaternion parameters. Hess, however, makes no direct use of our parameters and probably was not aware of the formula, $\zeta = \dfrac{\alpha Z + \beta}{\gamma Z + \delta}$, which lies at the basis of our discussion.

LECTURE III

In the lecture of yesterday we reached the conclusion that our parameters α, β, γ, δ may be expressed as quotients of simple σ-functions of the time t, and we now turn to the geometrical interpretation of these formulas.

As I have already asked you to notice, α, β, γ, δ are not ordinary elliptic functions of t, but functions which are affected by an exponential factor when t is increased by a period; in consequence of which I called them "multiplicative elliptic functions." When t is increased by the period $2\omega_1$, they are affected by an imaginary factor of the form $e^{i\psi}$, and when t is increased by the period $2i\omega_2$, by a real factor of the form κ.

Let us first of all consider *the curve described by the apex of the top on the fixed sphère.* This is the point $Z = \infty$ of the movable sphere, so that, reverting to the formula:

$$\zeta = \frac{\alpha Z + \beta}{\gamma Z + \delta},$$

it is obvious that the equation of the curve is

$$\zeta = \frac{\alpha}{\gamma} = ke^{\lambda t}\,\frac{\sigma(t + ia)}{\sigma(t - \omega_1 + ib)}.$$

Like α, β, γ, δ, this ζ is defined in terms of t by a multiplicative elliptic function of the first degree, involving besides the exponential factor only the quotient of two simple σ-functions.

This is an essential simplification of the representations of this motion given hitherto. Thus, were one to apply the methods used by Hermite in his *Applications des fonctions elliptiques*, published twenty years ago, and start not from the equation of ζ in terms of Z, but from those of $x+iy$, $-z$, $-x+iy$ in terms of $X+iY$, $-Z$, $-X+iY$ (see page 8), one would obtain for the motion of the apex of the top (whose coordinates are 0, 0, 1), the equation

$$x+iy=-2\,\alpha\beta,$$

which represents the motion by means of a multiplicative elliptic function of the second order. The curve thus defined is not the curve traced by the apex on the fixed sphere, but the orthogonal projection of this curve on the xy-plane.

I shall, for convenience, call curves like those which we have just been considering "multiplicative elliptic curves," distinguishing when necessary between those on the sphere and those on the plane, and assigning to them a degree corresponding to the number of simple σ-quotients in the expressions which define them. Thus the curve traced by the apex of the top on the fixed

sphere is a multiplicative elliptic curve of the first degree, its orthogonal projection one of the second degree. The earliest example of such a curve of the first degree is the herpolhode of a Poinsot motion, the motion of a body about its centre of gravity. That this herpolhode is such a curve was first shown by Jacobi.*

It is easy to get a notion of the geometrical character of the curve traced by the apex of the top. For the particular case when $l - ne_2 = 0$, the stereographic projection of the curve has the shape indicated in the following figure:

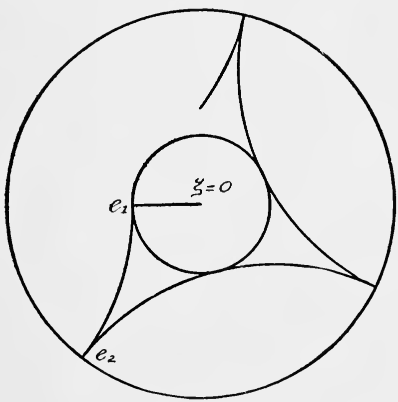

FIG. 5.

* Concerning the multiplicative elliptic curves, see Miss Winston's dissertation: *Ueber den Hermiteschen Fall der Laméschen Differentialgleichung*, Göttingen, 1897.

As we wish to restrict t to real values, we here make e_1 the lower limit of integration of the integral $t = \int \frac{du}{\sqrt{U}}$, or what comes to the same thing, suppose the t of the preceding formulas replaced by $t' = t + i\omega_2$.

The radius of the circle marked $u = e_1$ is the modulus of those points ζ for which t is 0, $2\,\omega_1, \cdots$; for all these points $u = e_1$. On the other hand, the radius of the circle marked $u = e_2$ is the modulus of those points ζ for which $t = \omega_1$, $3\,\omega_1, \cdots$.

The curve of the figure is that traced by the stereographic projection of ζ as t varies through real values, and consists of an infinite number of congruent arcs which touch the inner circle and form cusps at the outer one. If the top be given an initial thrust sideways (when $l - ne_2$ is no longer 0), these cusps will be replaced by loops or wave-crests.

Evidently any one of these arcs may be brought into coincidence with the consecutive one by one and the same rotation about the origin. The transformation which effects this rotation is $\zeta' = e^{i\psi_0}\zeta$, so that the meaning of the imaginary factor $e^{i\psi_0}$, by which ζ is affected when t is increased by the real period $2\,\omega_1$ is perfectly obvious. We shall find that since ζ is affected by the real

factor κ when t is increased by the imaginary period $2\,i\omega_2$, the effect on the curve of this increase in t is to transform it into a curve similar to itself, and symmetrically placed with respect to the origin.

But before attempting a more minute examination of the curve traced by the apex of the top, let us consider *the polhode and herpolhode of the motion.*

On each instantaneous axis of rotation let a segment be measured from the fixed point, equal in sense and magnitude to the amount of rotation about this axis. The aggregate of these segments constitute a portion of one cone if they be caused to remain fixed in the moving body, of another if they be caused to remain fixed in space. The first cone, or the curve in which its elements terminate, is called the "polhode," the second the "herpolhode," and it is evident that the motion of the body may be had by rolling the first cone or curve on the second cone or curve.

To obtain the equation of the polhode, consider the infinitesimal rotation in time dt about the axis for which the components of rotation with respect to X, Y, Z, are p, q, r, respectively. The axis is for the instant fixed in space, and we have for the effect of the rota-

tion on any point of the moving sphere the equations:

$$X = + X' \quad - rdtY' + qdtZ',$$
$$Y = + rdtX' + Y' \quad - pdtZ',$$
$$Z = - qdtX' + pdtY' + Z',$$

For this motion therefore the quaternion parameters (see page 4) are:

$$A' = \frac{p}{2}dt, \quad B' = \frac{q}{2}dt, \quad C' = \frac{r}{2}dt, \quad D' = 1,$$

and therefore the corresponding parameters $\alpha, \beta, \gamma, \delta$ are:

$$\alpha' = 1 + \frac{ir}{2}dt, \qquad \beta' = \frac{-q + ip}{2}dt,$$

$$\gamma' = \frac{q + ip}{2}dt, \qquad \delta' = 1 - \frac{ir}{2}dt.$$

If therefore α, β, γ, δ (unprimed) be the parameters of the transformation from the axes X, Y, Z fixed in the body to the axes x, y, z fixed in space, we may obtain the parameters $\alpha + d\alpha$, $\beta + d\beta$, $\gamma + d\gamma$, $\delta + d\delta$ of the transformation which defines the position of the body after the infinitesimal rotation, by combining the two substitutions:

$$\zeta_1 = \alpha\zeta_1' + \beta\zeta_2', \qquad \zeta_1' = \alpha' Z_1 + \beta' Z_2,$$
$$\zeta_2 = \gamma\zeta_1' + \delta\zeta_2', \qquad \zeta_2' = \gamma' Z_1 + \delta' Z_2,$$

the result of which is:

$$\zeta_1 = (\alpha\alpha' + \beta\gamma')Z_1 + (\alpha\beta' + \beta\delta')Z_2,$$
$$\zeta_2 = (\gamma\alpha' + \delta\gamma')Z_1 + (\gamma\beta' + \delta\delta')Z_2.$$

It follows, therefore, that

$$\alpha + d\alpha = \alpha\alpha' + \beta\gamma', \qquad \beta + d\beta = \alpha\beta' + \beta\delta',$$
$$\gamma + d\gamma = \gamma\alpha' + \delta\gamma', \qquad \delta + d\delta = \gamma\beta' + \delta\delta',$$

whence
$$d\alpha = \left(\frac{ir}{2}\alpha + \frac{q+ip}{2}\beta\right)dt,$$
$$d\beta = \left(\frac{-q+ip}{2}\alpha - \frac{ir}{2}\beta\right)dt,$$
$$d\gamma = \left(\frac{ir}{2}\gamma + \frac{q+ip}{2}\delta\right)dt,$$
$$d\delta = \left(\frac{-q+ip}{2}\gamma - \frac{ir}{2}\delta\right)dt.$$

Whence finally:
$$p + iq = 2\,i\left(\beta\frac{d\delta}{dt} - \delta\frac{d\beta}{dt}\right),$$
$$-p + iq = 2\,i\left(\alpha\frac{d\delta}{dt} - \delta\frac{d\alpha}{dt}\right),$$
$$r = 2\,i\left(\alpha\frac{d\delta}{dt} - \gamma\frac{d\beta}{dt}\right).$$

We will not stop to derive the corresponding equations for the components π, κ, ρ of the herpolhode. They differ from those just obtained for p, q, r only in having α and δ interchanged and the signs of β and γ changed.

But I wish to make two remarks which are suggested by the above reckoning, with regard to the usefulness of our parameters α, β, γ, δ. The one is that two linear substitutions in terms of them combine binarily instead of quaternarily as do the

corresponding quaternion substitutions; the other, that the four linear differential equations which define them in terms of t, p, q, r break up into two pairs, in one of which only α and β are involved, in the other only γ and δ. To appreciate how important this advantage is, one need only compare with our discussion the discussion of the same question in Darboux's *Leçons sur la théorie générale des surfaces.*

Returning to our spherical top, and substituting in the general equations which we have just obtained the values of $\alpha, \beta, \gamma, \delta$ which characterize its motion, we have for its polhode and herpolhode not equations of the second degree, as was to have been expected from the expressions for $p + iq$, etc. in terms of $\alpha, \beta, \gamma, \delta$, but much simpler expressions. I cannot give the reckoning which leads to them since I have not given the values of the constants $k_1, \lambda_1, \cdots$ which appear in the formulas on page 31. But the expressions themselves are of the form

$$p + iq = k' e^{\lambda' t} \frac{\sigma(t + \omega_1 - ia - ib)}{\sigma(t)}, \quad r = n;$$

$$\pi + i\kappa = k'' e^{\lambda'' t} \frac{\sigma(t + \omega_1 - ia + ib)}{\sigma(t)}, \quad \rho = l.$$

Both the polhode and the herpolhode of the spherical top are elliptic plane curves of the first degree. Darboux has given this result in his edition of Despeyrous' Mechanics, obtaining it by the use of

elliptic integrals instead of elliptic functions. He does not call the curves elliptic curves of the first degree, but curves of the same character as the herpolhode of a Poinsot motion. It should be added that the curves are of the first degree in the case of the *spherical* top only.

Our theorem is closely connected with the celebrated theorem of Jacobi already mentioned: that the motion of the top may be represented by the relative motion of two Poinsot motions (or rotations about the centre of gravity); for both the polhode and herpolhode of the top's motion are themselves herpolhodes of Poinsot motions, being elliptic curves of the first degree. One may demonstrate Jacobi's theorem most simply by expressing the α, β, γ, δ of each of the Poinsot motions in terms of t, and then combining the two motions.

I may finish this part of my discussion with the remark that the attention of students of the geometry of Salmon and Clebsch is apt to be confined too exclusively to algebraic curves. We have before us an illustration of the value of transcendental curves. It is only in the very exceptional case when the multiplicative factor $\kappa = 1$, and ψ_0 is commensurable with π, that the curves we have been studying become algebraic.

To sum up the conclusions which we have thus far established; we have proved that *the motion of*

the spherical top on a fixed point of support may be completely defined geometrically in terms of elliptic curves of the first degree. We have also shown that the variation of the parameters $\alpha, \beta, \gamma, \delta$ with the time t may be pictured by curves of the same character.

Let us now resume the study of the *curve traced by the apex of the top.*

The parameters α, β, γ, δ, and ζ are all elliptic functions of the argument t, and the full meaning of elliptic functions comes to light only when the argument is supposed capable of taking complex values. Thus only, in particular, will the double periodicity of the functions come into evidence. There exists, then, an analytical necessity, so to speak, that we complete our geometrical study of the top's motion by extending it to complex values of t. When that has been accomplished, I shall show that to the entire aggregate of possible motions of the top in complex time there corresponds the free motion of a certain rigid body in non-Euclidean space, and thus bring to a definite outcome the considerations which I presented at the close of my first lecture.

Our problem being to determine the path traced by the point ζ when the point t is made to describe *any* path in the t-plane, it is clearly of prime importance that we determine first of all the image on the ζ-sphere of a parallelogram of periods in the

t-plane. To that, indeed, we shall confine our attention. Instead, however, of finding this image directly we shall find it easier to obtain the images of the four half-sheets of the Riemann surface of $\sqrt{U}$, of which, it will be remembered, the four smaller rectangles into which the entire parallelogram of periods was subdivided were severally the images.

Let us first reproduce (in Fig. 6) the figure of the parallelogram of periods (see page 29) and that of the Riemann surface of $\sqrt{U}$.

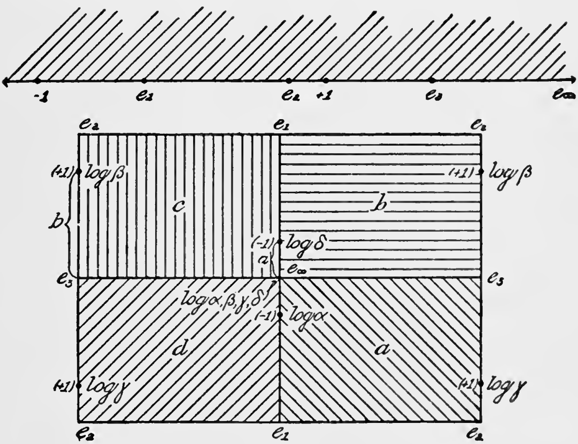

Fig. 6.

I have given different markings to all four rectangles in order to be able to distinguish readily between their several images in the figure which

we are to construct. It will be remembered (see page 28) that $\log \alpha$ and $\log \gamma$ became infinite at the points $u = -1$ and $u = +1$, respectively, of one sheet of the $\sqrt{U}$-surface, and that $\log \beta$ and $\log \delta$ became infinite at the corresponding points of the other sheet — the other functions in each case remaining finite. In the figure, a and d are the images of the positive and negative halves of the first of these sheets, and c and b the images of the positive and negative halves of the second.

Our ζ is expressed in terms of u by the elliptic integral of the third kind:

$$\log \zeta = \log\left(\frac{\alpha}{\gamma}\right) = \int \frac{-\sqrt{U} + i(nu - l)}{u^2 - 1} \cdot \frac{du}{\sqrt{U}}.$$

We may now draw the following conclusions immediately:

$\frac{d}{du} \log\left(\frac{\alpha}{\gamma}\right)$ is complex along the segments e_1e_2, e_3e_∞ of the real axis of the u-plane, but real along the segments e_2e_3, $e_\infty e_1$. Therefore $\frac{\alpha}{\gamma}$ or ζ moves along a meridian of the ζ-sphere when u moves along the real axis from e_2 to e_3 or from e_∞ to e_1; but, on the other hand, describes one of the arcs which appeared in the figure of the real motion of the top's apex, when u moves on the real axis from e_1 to e_2, and an arc different from this, when u moves from e_3 to e_∞.

Again, $\frac{d}{du}\log\left(\frac{u}{\gamma}\right)$ vanishes when $u = e_\infty$, in the first approximation as $\frac{1}{u^2}$, and takes the finite value $\frac{1}{1 - e_2^2}$ when $u = e_2$ (this because of the hypothesis which we retain here, that $l - ne_2 = 0$); when $u = e_1$ or e_3, on the other hand, it becomes infinite, as $(u - e_1)^{-\frac{1}{2}}$ or $(u - e_3)^{-\frac{1}{2}}$. Therefore the curve traced by the point ζ as the point u moves along the real axis from e_∞ through e_1, e_2, e_3, to e_∞ will present angles whose measure is π at the points corresponding to e_∞ and e_2, and angles whose measure is $\frac{\pi}{2}$ at the points corresponding to e_1 and e_3.

I will not give the image of the $\sqrt{U}$-surface on the sphere, but the stereographic projection of this image on the xy-plane from the point $\zeta = \infty$. If to the explanations already given it be added that ζ, whose value in terms of t is $ke^{\lambda t}\frac{\sigma(t + ia)}{\sigma(t - \omega_1 + ib)}$, becomes 0 and ∞ respectively at the points -1 and $+1$ of the contour of the half-sheet or rectangle a, and remains finite and different from 0 for all points on the contour of b, it will readily be seen that the images of the half-sheets or rectangles a, b, are roughly of the form indicated in the following figure: the two contours which we have marked $e_\infty\ e_1\ e_2\ e_3\ e_\infty$ being the stereographic projections of images of the real u-axis first when this axis is

regarded as the contour of the positive half-sheet a, second when it is regarded as the contour of the negative half-sheet b.

The two arcs $e_1 e_2$ are similar and symmetrically placed with respect to the point $\zeta = 0$. The one which lies to the left appeared in the figure of the real motion of the top's apex (Fig. 5).

If now we complete this figure by a second half symmetrical with this first half with respect to the horizontal axis $e_1 e_\infty$, we obtain the image of the entire

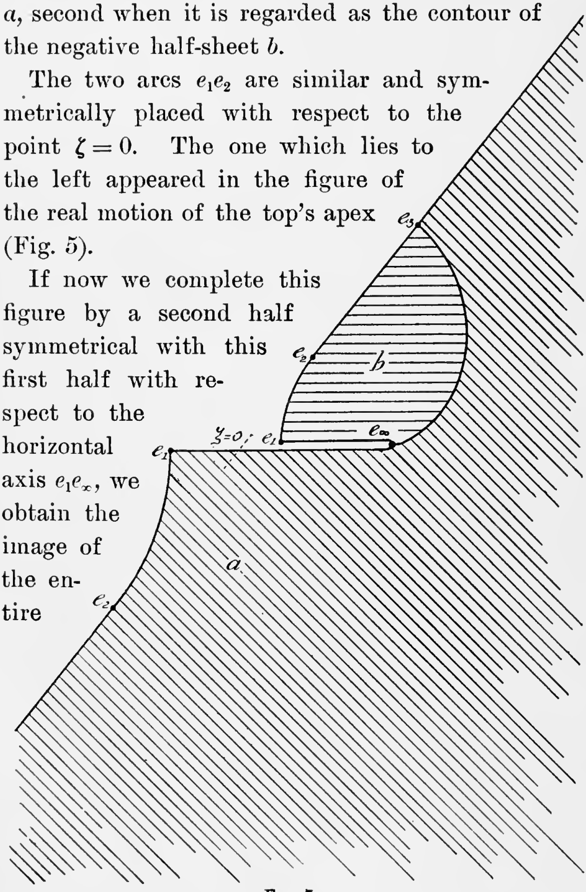

FIG. 7.

$\sqrt{U}$-surface or of the entire parallelogram of periods in the t-plane (Fig. 8). We suppose an incision made in the $\sqrt{U}$-surface along the segment $e_1 e_2 e_3$ of the real axis.

It will be noticed that the image covers doubly the portion of the plane which lies within the two arcs e_1, e_2, e_3, $\zeta = \infty$, which lie to the right, the two sheets being joined along a branch line which runs from e_∞ to $\zeta = \infty$. From the figure we infer that e_∞ is a branch point of t, but not so the point $\zeta = \infty$; for a circuit cannot be made of the point $\zeta = \infty$ without passing into the portion of the plane bounded by the half-arcs e_1, e_2, $\zeta = 0$, lying to the left, which does not belong to the image. And these conclusions may readily be verified by reckoning.

We may describe our figure as a quadrilateral, one of whose pairs of opposite sides are the rectilineal segments running from the points e_2, *through* $\zeta = \infty$, *and which, were they produced, would intersect at* $\zeta = 0$, *and the other pair, the two curvilinear arcs* $e_2 e_1 e_2$.

The sides of each pair go over into each other by the substitution of ζ which corresponds to a change of t by one of the periods $2\,\omega_1$, $2\,i\omega_2$: the straight sides by the rotation about $\zeta = 0$ defined by the "elliptic substitution" $\zeta' = e^{i\psi_0}\zeta$, which we have already considered and which we have indicated in the figure by the double-headed curved arrows; the curved sides by the transformation

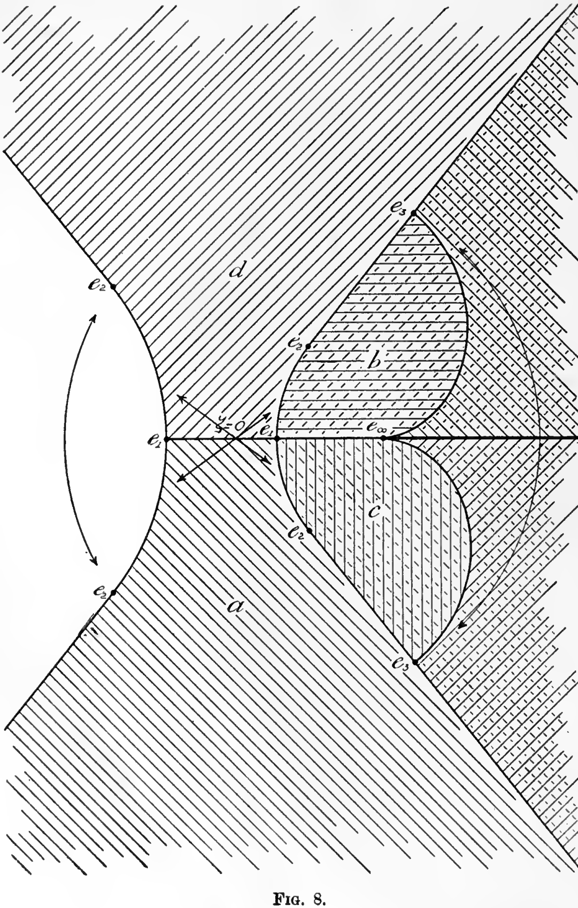

Fig. 8.

defined by the "hyperbolic substitution" $\zeta' = \kappa\zeta$, in consequence of which they are similar and symmetrically placed with respect to the centre of similitude $\zeta = 0$. In the figure we have indicated the latter transformation by the double-headed straight arrows which intersect at $\zeta = 0$. The significance of both the periods $2\,\omega_1$, $2\,i\omega_2$ for the curve traced by the apex of the top is thus made evident by our figure. And indeed we have now clearly before us for the first time the reason that the curve described in real time should be represented by elliptic functions. It is but a portion of the complete curve, or rather domain, which comes to light when we avail ourselves of the entire field of complex numbers in which the representation of both periods is alone possible.

The Riemann surface determined by $\zeta = \dfrac{\beta(t)}{\delta(t)}$, the curve traced by the opposite extremity of the top's axis, $Z = 0$, may be constructed similarly.

For real values of t we have $\dfrac{\beta(t)}{\delta(t)} = -\dfrac{\bar{\gamma}(t)}{\alpha(t)}$, which means simply that $\dfrac{\alpha(t)}{\gamma(t)}$ and $\dfrac{\beta(t)}{\delta(t)}$ are opposite extremities of one and the same diameter of the sphere. For complex values of t this formula is to be replaced by the more general one

$$\frac{\beta(t)}{\gamma(t)} = -\frac{\bar{\gamma}(\bar{t})}{\bar{\alpha}(t)}.$$

If now we suppose these two Riemann surfaces to be projected back again to the surface of the fixed sphere, and the points of the two which correspond to the same value of t to be joined, *the resulting system of rays will represent the* ∞^2 *positions which the axis of the top may take in the general (non-Euclidean) motion which corresponds to any motion of t in the parallelogram of periods.*

Of these ∞^2 "axes," only those pass through the centre of the sphere which correspond to real values of t. These are the axes which meet the curved arc $e_2e_1e_2$ of the preceding figure which lies to the left. Those axes which meet the other curved arc $e_2e_1e_2$ intersect in another point of the central line (*i.e.* of the vertical through the centre of the sphere); namely, the point into which the centre of the sphere is transformed by the hyperbolic substitution already explained. A visible representation of the possible motions of the top's axis in complex time is to be had by constructing the figures for $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ on an actual sphere and joining a number of corresponding points by straight lines.

The doubly infinite systems of the rays which are elements of the polhodes and herpolhodes of all motions possible in complex time, may be con-

structed in like manner, and a complete geometrical representation be thus obtained of the top's motion. The constructions are more complicated, but there is no essential difficulty in carrying them out.

In fact, the only serious difficulty in this entire method of discussion is, that all our ordinary conceptions of mechanics involve the notion that time is capable of but one sort of variation. We are so accustomed to regard the mechanical conditions which correspond to small values of t, as, so to speak, the *cause* of those which correspond to greater values, and to picture the changes of configuration as following one another in definite order with the varying time, that we find ourselves at a loss for a mechanical representation when t, by being supposed complex, becomes capable of two degrees of variation.

To avoid this difficulty as far as possible, let us suppose t no longer capable of varying in every direction in the parallelogram of periods, but only along a line parallel to the real axis. In other words, in $t = t_1 + it_2$, let us regard t_2 as constant in each particular case, and t_1 as alone varying. In this manner, by subsequently giving t_2 all possible values, we may take into account all possible complex values of t, but we conceive them as ranged along the ∞^1 parallels to the real axis. Regarded thus,

the Riemann surfaces $\frac{\alpha}{\gamma}, \frac{\beta}{\delta}$ become carriers of certain curve systems, and the system of ∞^2 axes is distributed among ∞^1 ruled surfaces.

In this manner we separate the totality of the positions of the top in complex time into an infinite number of simply infinite sets of positions. These sets of positions are characterized not only by the initial values of t, but by the values of the constants of integration, which must have been introduced had the reckoning which we have merely sketched been actually carried out. It should perhaps have been stated earlier that in the interest of complete generality these constants must now be supposed complex, for we are now operating in the domain of complex numbers. Moreover, only by supposing them complex shall we have constants enough at our disposal to meet all the conditions of our generalized problem of motion.

So far our figures have been constructed with a view to obtaining a clear geometrical representation of the entire content of our analytical formulas. But their chief interest lies in this: that one can give them a *real* dynamical meaning, that one can find a real mechanical system by whose motions they may be generated. I assert that *one can determine a certain free mechanical system, namely, a rigid body freely moving in non-Euclidean space*

under the action of certain definite forces, which in real time carries out exactly that infinity of forms of motion which we have just been describing, the one or other of them according to the choice made of the initial conditions of motion. The mechanical system is a generalized one, but it belongs to the domain of real dynamics.

Let us consider the general problem of the motion of a rigid body under the action of any forces, in the non-Euclidian space whose absolute is the surface:

$$x^2 + y^2 + z^2 - t^2 = 0.$$

The earliest investigation of the motion of a rigid body in non-Euclidean space was made by Clifford in 1874 — though the investigation was not published until after his death, in his collected works. The same problem has been considered also by Heath in the *Philosophical Transactions*, 1884. Both these mathematicians, however, have treated the case of the elliptic non-Euclidean geometry, not the hyperbolic, and have contented themselves with establishing the differential equations of the problem.

I shall proceed analytically, as this method is more readily understood by one who is not well versed in non-Euclidean geometry, and immediately obtain differential equations for the motion of a certain rigid body in non-Euclidean space perfectly

analogous to the equations for the motion of the top in real time, but involving two sets of variables.

To have the general case before us at once, I suppose the parameters ϕ, ψ, ϑ, and the time t, all complex and set

$$\phi = \phi_1 + i\phi_2, \quad \psi = \psi_1 + i\psi_2, \quad \vartheta = \vartheta_1 + i\vartheta_2, \quad t = t_1 + it_2.$$

These parameters are connected with T and V, the kinetic and potential energy, by the well-known Lagrange equations:

$$\frac{d\left(\frac{\delta T}{\delta \vartheta'}\right)}{dt} = \frac{\delta(T - V)}{\delta \vartheta}, \quad \frac{d\left(\frac{\delta T}{\delta \phi'}\right)}{dt} = \frac{\delta(T - V)}{\delta \phi},$$

$$\frac{d\left(\frac{\delta T}{\delta \psi'}\right)}{dt} = \frac{\delta(T - V)}{\delta \psi}.$$

In these equations set

$$T = T_1 + iT_2, \quad V = V_1 + iV_2.$$

Since, then, $$\frac{\delta T}{\delta \vartheta'} = \frac{\delta T_1}{\delta \vartheta'_1} + i\frac{\delta T_2}{\delta \vartheta'_1}$$

$$= \frac{\delta T_1}{\delta \vartheta'_1} - i\frac{\delta T_1}{\delta \vartheta'_2};$$

and similarly,

$$\frac{\delta T}{\delta \phi'} = \frac{\delta T_1}{\delta \phi'_1} - i\frac{\delta T_1}{\delta \phi'_2} \text{ and } \frac{\delta T}{\delta \psi'} = \frac{\delta T_1}{\delta \psi'_1} - i\frac{\delta T_1}{\delta \psi'_2};$$

and since, furthermore, by our hypothesis, $dt = dt_1$, the first of our equations breaks up into the two equations involving real variables only,

$$\frac{d\left(\frac{\delta T_1}{\delta \vartheta'_1}\right)}{dt_1} = \frac{\delta(T_1 - V_1)}{\delta \vartheta_1}, \quad \frac{d\left(\frac{\delta T_1}{\delta \vartheta'_2}\right)}{dt_1} = \frac{\delta(T_1 - V_1)}{\delta \vartheta_2};$$

and the remaining two equations behave similarly.

Thus, every real mechanical problem again reduces to a real problem when the variables are made complex, provided the real part only of the complex t be supposed to vary, but the problem of a motion involving twice the number of variables.

Applying this general conclusion to the particular question before us, it is evident without any further discussion that the problem of the motion in complex time of a top whose point of support is fixed is changed into a problem of real dynamics; the problem of the non-Euclidean motion of a rigid body. This motion has six degrees of freedom instead of three, corresponding to the six parameters, ϑ_1, ϑ_2, ϕ_1, ϕ_2, ψ_1, ψ_2, and its kinetic and potential energy are T_1 and V_1, the real parts of the complex T and V.

But what is the rigid body, and what the force producing the motion? We shall content ourselves with simply answering these questions without entering upon the considerations appertaining to non-

Euclidean geometry by which our conclusions are reached.

The equation of the absolute being

$$x^2 + y^2 + z^2 - t^2 = 0,$$

the integral

$$\int \frac{(ux + vy + wz + \omega t)^2}{(u^2 + v^2 + w^2 - \omega^2)(x^2 + y^2 + z^2 - t^2)}\, dm,$$

evaluated throughout any body in the corresponding non-Euclidean space, is called the "second moment" of the body with respect to the plane whose co-ordinates are u, v, w, ω. *In the particular case before us this integral, when evaluated, will be equal to* 1, *independently of the values of* u, v, w, ω.

Remembering that u, v, w, ω are constants with respect to the integration, the result may be written

$$\frac{Au^2 + 2\,Buv + \cdots}{u^2 + v^2 + w^2 - \omega^2}, \text{ which therefore} = 1.$$

Now the surface whose equation in tangential co-ordinates is

$$Au^2 + 2\,Buv + \cdots = 0$$

is called the "null-surface." In the case before us, therefore, *the null-surface coincides with the absolute.* This is the rigid body of our non-Euclidean motion.

The force producing the motion may be defined as follows: In the figure (Fig. 9) let g represent the fixed axis of gravitation (through the point of support

of the top), r the axis of the top, and p the non-Euclidean perpendicular common to g and r. The angle between g and r is then defined as $\vartheta = \vartheta_1 + i\vartheta_2$, where ϑ_1 represents the angle between the planes gp and rp, and $i\vartheta_2$ in non-Euclidean angular measure is the distance p.

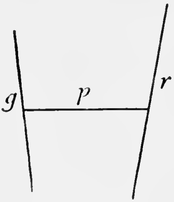

Fig. 9.

The force is then the wrench represented in intensity by $P \sin \vartheta$, of which the real part represents the rotating force acting about p and the imaginary part represents the thrust along p.

In conclusion, allow me to remark once again that this non-Euclidean geometry involves no metaphysical consideration, however interesting such considerations may be. *It is simply a geometrical theory which groups together certain geometrical relations in real space in a manner peculiarly adapted to their study.*

LECTURE IV

In the latter part of yesterday's lecture we ventured a little way into what Professor Newcomb has called the "fairyland of mathematics." Ignoring the limitation of the top's motion to real time, we gave full play to our purely mathematical curiosity. And there can be no doubt that it is proper and indeed necessary within due limits to proceed after this manner in all such investigations as that now before us. It is possible only thus to develop a strong and consistent mathematical theory. But we should not yield ourselves wholly to the charm of such speculations, but rather control them by being ever ready to return to the actual problems which nature herself proposes.

We turn again to-day, therefore, to the real top, and proceed to investigate its motion when the point of support is no longer fixed, but movable in the horizontal plane. This is the case of the *ordinary toy top.*

It has been well known since the time of Poisson that the differential equations of this motion can be integrated in terms of the hyperelliptic inte-

grals. And it is the main purpose of my present lecture to show *that these integrals may be treated in a manner quite analogous to that in which the elliptic integrals were treated, by aid of the general "automorphic functions," of which the elliptic functions are a special class.*

The "toy top" has five degrees of freedom of motion, two of them relating to the horizontal displacement of the centre of gravity, and the other three to the motion around this centre. The horizontal motion of the centre of gravity is very simple, being, as is well known, a rectilinear motion of constant velocity. Consequently, no essential restriction of the problem is involved in assuming the horizontal projection of the centre of gravity to be a fixed point. By this assumption the problem is again reduced to one of three degrees of freedom only, and we have besides t no other variables to consider than the parameters ϕ, ψ, ϑ or α, β, γ, δ of the previous discussion — the parameters here defining the position of the top with respect to axes through its centre of gravity.

To obtain first the ordinary formulas which define the motion in terms of the astronomical parameters: let G represent the weight of the top, s the distance of its centre of gravity from the point of support, and again represent the product Gs, *i.e.* the static moment, by P. Also, for the sake of sim-

plicity, let us again suppose that the three principal moments of inertia of the top, in this case with respect to the axes through its centre of gravity, are all equal to 1.

Then the kinetic energy, T, and the potential energy, V, are given by the following equations: viz.

$$T = \tfrac{1}{2}(\phi'^2 + \psi'^2 + 2\,\phi'\psi' \cos\vartheta + \vartheta'^2 + Ps \sin^2\vartheta \cdot \vartheta'^2),$$

$$V = P \cos\vartheta,$$

which differ from the corresponding expressions in the special case where the point of support is fixed only in the appearance of the additional term $Ps \cdot \sin^2\vartheta \cdot \vartheta'^2$ in T. As this term will disappear if $s = 0$, though we take Gs, *i.e.* P, different from zero, the elementary case may be described from the present point of view as that of a top of infinite weight whose centre of gravity coincides with its point of support.

On substituting these values for T and V in the first two Lagrange equations,

$$\frac{d\dfrac{\delta T}{\delta\phi'}}{dt} = 0, \quad \frac{d\dfrac{\delta T}{\delta\psi'}}{dt} = 0,$$

we obtain immediately, as before, the two algebraic first integrals

$$\phi' + \psi' \cos\vartheta = n,$$

$$\psi' + \phi' \cos\vartheta = l.$$

If from these last equations we reckon out ϕ' and ψ', and substitute the resulting values in the integral of energy

$$T + V = h,$$

we obtain t, ϕ, and ψ in the form of integrals in terms of the variable ϑ.

As before, we set $u = \cos\vartheta$, and

$$U = 2\,Pu^3 - 2\,hu^2 + 2\,(ln - P)\,u + (2\,h - l^2 - n^2),$$

when these integrals become

$$t = \int \frac{du\sqrt{(1 + Ps) - Psu^2}}{\sqrt{U}},$$

$$\phi = \int \frac{n - lu}{1 - u^2} \cdot \frac{du\sqrt{(1 + Ps) - Psu^2}}{\sqrt{U}},$$

$$\psi = \int \frac{l - nu}{1 - u^2} \cdot \frac{du\sqrt{(1 + Ps) - Psu^2}}{\sqrt{U}}.$$

These formulas differ from the corresponding formulas for the elementary case in that the new irrational factor $\sqrt{(1 + Ps) - Psu^2}$ here appears in the numerator of each integrand. In consequence, we have now to do with hyperelliptic integrals, $p = 2$. In addition to the former branch-points of the Riemann surface in the u-plane, viz. e_1, e_2, e_3, e_∞, two new real branch-points appear, viz.:

$$u = \pm\sqrt{\frac{1 + Ps}{Ps}}.$$

I shall call them e_4, e_6, and assume them to be numerically greater than e_3. The Riemann surface is therefore a surface of two sheets with six branch-points e_1, e_2, e_3, e_4, e_∞, e_6, ranged along the real axis of the u-plane, as indicated in the following figure:

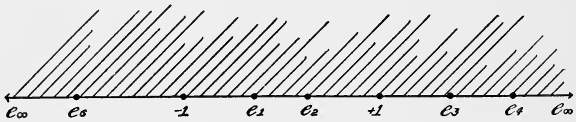

FIG. 10.

In addition to the branch-points, I have indicated the positions of the points $+1$, -1, since these particular values of u, corresponding to $\vartheta = 0$, $\vartheta = \pi$, play, as in the elementary case, a special rôle in our discussion.

The time t is no longer an integral of the first kind; that is to say, an integral which remains finite for all values of u, but an integral of the second kind, which becomes infinite for $u = \infty$, as $\sqrt{-2su}$. An integral of the second kind, it may be added, is one having a point of algebraic discontinuity only. The integrals ϕ and ψ, on the other hand, have each of them, as before, four logarithmic points of discontinuity; namely, the four points $u = \pm 1$ of the Riemann surface.

The first step to be taken is to replace the integrals ϕ and ψ by *normal* hyperelliptic integrals

of the third kind; that is, by integrals possessing each but *two* logarithmic points of discontinuity with the residues $+1$ and -1. This is accomplished precisely as in the elementary case, by introducing $\log \alpha$, $\log \beta$, $\log \gamma$, $\log \delta$. As before, these prove to be normal integrals of the third kind, each having a logarithmic discontinuity (with the residue $+1$) at one of the points $u = \pm 1$, and all having a second logarithmic discontinuity in common (with the residue -1) at the point $u = \infty$. This follows at once from the result of the reckoning if it be noticed that the expression $(1 + Ps) - Psu^2$ reduces to 1 for $u = \pm 1$.

It is evident, therefore, that *the parameters* α, β, γ, δ *play the same fundamental rôle here as in the case of the top whose point of support is fixed.* And in the following discussion we shall no longer use ϕ and ψ, but α, β, γ, δ. These variables possess on the Riemann surface a 0-point each at one of the four points $u = \pm 1$, and a common ∞-point at $u = \infty$. I have not thought it necessary to enter into the details of this reduction, as it is so completely analogous to the reduction in the more elementary case.

But when we attempt to repeat the next step of the previous discussion, and endeavor, by inverting the hyperelliptic integral t, to assign to t the rôle of independent variable, we find at

once that there is a profound difference between our present problem and the previous more special problem. This difference is masked when we confine our attention to the top's motion in real time. For as t varies, remaining always real, the value of u vibrates as before between the values e_1 and e_2, while ϕ and ψ are each increased by real periods. The difference comes to light, however, as soon as, allowing t to take complex values, we proceed to construct in the t-plane the image of the Riemann surface. As the image of a half-sheet of this surface, we have now, instead of the simple rectangle of the elementary case, an open hexagon with one of its angular points at infinity, as in the following figure:

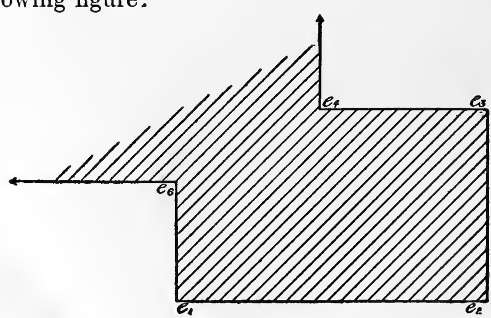

Fig. 11.

and when by the methods of symmetrical and congruent reproduction, we go on to construct from this figure the image of the entire Riemann sur-

face, we at once encounter the difficulty that this image will cover the t-plane not simply, as in the elementary case, but rather with an infinite number of overlapping hexagonal pieces. To a single point in the t-plane, therefore, will correspond not one, but an infinite number of values of u, that is to say, u is no longer a *uniform* function of t.

I may remark that it is often said that the inversion of the hyperelliptic integrals is impossible. This is not true; it is not impossible to invert them, but to get uniform functions by the process.

There is a well known method of generalizing the result of inverting the elliptic integrals and obtaining functions, "hyperelliptic functions," as they are called, which are in a proper sense the generalization of the elliptic functions. The method is due to Jacobi, and goes by his name.

There are two hyperelliptic integrals of the first kind in the case before us:

$$v_1 = \int \frac{du}{\sqrt{U} \cdot \sqrt{1 + Ps - Psu^2}},$$

$$v_2 = \int \frac{u \cdot du}{\sqrt{U} \cdot \sqrt{1 + Ps - Psu^2}}.$$

Jacobi forms double ϑ-functions of v_1, v_2, viz. $\theta(v_1, v_2)$, in terms of which he seeks to express the other variables as uniform functions. This

is, perhaps, the greatest achievement of Jacobi, and for general investigations of the highest importance, but it promises us little aid in the problem which we are considering. To avail ourselves of it, we should need first to develop a method for determining what values of v_1, v_2 correspond to the same value of t. We are therefore reduced to the direct computation of hyperelliptic integrals if we wish to avoid the complicated equation for v_1 and v_2 which results if we eliminate t.

Is it possible, then, by any means whatsoever, to obtain for the general motion of the top formulas analogous to those which we succeeded in establishing for the top whose point of support was fixed? Yes, by availing ourselves of the *theory of the uniform automorphic functions.*

A uniform automorphic function of a single variable η is a function $f(\eta)$, which satisfies the functional equation

$$f\left(\frac{a_\nu \eta + b_\nu}{c_\nu \eta + d_\nu}\right) = f(\eta),$$

where a_ν, b_ν, c_ν, d_ν have given constant values for each of the values of ν: $1, 2, 3 \cdots \infty$ — for all of which the functional equation is satisfied.

The automorphic functions, therefore, are functions which are transformed into themselves by an infinite but discontinuous group of linear substitu-

tions. They are the generalization of the elliptic functions which consists in generalizing the periodicity of these functions, but leaving the number of the variables unchanged, while Jacobi's hyperelliptic functions are a generalization which consists in increasing the number of variables, but leaving the periodicity unchanged.

I shall present what I have to say regarding them geometrically. And, indeed, the general notion of these automorphic functions, as well as the knowledge of their most important properties, originated from geometrical considerations, and geometrical considerations only. Even now the analytical details of the theory have been only partially developed.

Our problem, as we are now to conceive of it, is this: *to define a variable η, of which t, α, β, γ, δ shall be uniform automorphic functions*, as were α, β, γ, δ of t itself in the elementary case.

To revert to the elementary case — the fact that t was itself a "uniformizing" variable, *i.e.* a variable of which u was a uniform function, was brought to light by finding that when the image in the t-plane of a single half-sheet of the Riemann surface on the u-plane was reproduced by symmetry and congruence, this image covered the t-plane *simply*. May we not, then, construct in the plane of a variable η a rectangular hexagon which shall be the image

in the η-plane of a half-plane u, and which on being reproduced shall cover the η-plane or a portion of it simply, and then subsequently, from a study of the conditions which determine this hexagon, derive in definite analytical form the functional relation between η and u?

It is in fact possible, as the theory of automorphic functions shows, to construct such a rectangular hexagon, and that in essentially but one way. Its sides are not line segments, but arcs of circles which themselves cut the real axis of the η-plane at right angles. It has the following form:

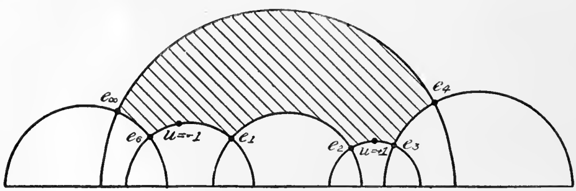

FIG. 12.

The mere geometrical requirement that the figure be made up of arcs of circles which cut the real axis orthogonally, and cut each other orthogonally also at the six points e_1, e_2, e_3, e_4, e_∞, e_6, is of course not enough to determine it completely. There are a certain number of parameters which remain undetermined, and which are to be so determined that the hexagon is an actual conformal representation

of the half u-plane with the given branch-points $e_1, e_2 \cdots e_6$. The fundamental theorem of the theory of automorphic functions declares that this can be accomplished in one, and essentially but one, way.

Having determined the image of the one half-sheet of the Riemann surface on the u-plane, the infinitely many remaining images are to be had by constructing the figure into which the original image is transformed by inversion with respect to each circle of which one of its sides is an arc, by repeating the same construction for the resulting hexagons, and so on indefinitely.

By this process the entire upper half of the η-plane is simply covered without overlapping by rectangular hexagons, whose sides are circular arcs. Each of these hexagons is an image of a half-sheet of the Riemann surface. And if they be alternately shaded and left blank, the shaded ones are images of positive half-sheets, the blank ones of negative half-sheets of the surface.

Evidently, then, to a single point in the η-plane there corresponds but a single point in the Riemann surface, or *u and $\sqrt{U}$ are uniform functions* of η. On the other hand, the points in two of the hexagons which correspond to the same value of u, $\sqrt{U}$, and may be called "equivalent points," are connected by a formula of the form $\eta' = \dfrac{a_\nu \eta + b_\nu}{c_\nu \eta + d_\nu}$, as in

the special elliptic case the corresponding points of two of the parallelograms of periods were connected by the formula $t' = t + 2m_1\omega_1 + 2m_2 i\omega_2$. Thus u and $\sqrt{U}$ are uniform automorphic functions of η, satisfying the equation:

$$f(\eta) = f\left(\frac{a_\nu \eta + b_\nu}{c_\nu \eta + d_\nu}\right).$$

I may remark that Lord Kelvin made use of this sort of symmetrical reproduction more than fifty years ago in his researches on electrostatic potential. But his figures were solids bounded by portions of spherical surfaces, and his aim was so to determine these that only a finite number of other distinct solids should result from them by the process of reproduction.

Not only u and $\sqrt{U}$, but also $\sqrt{1 + Ps - Psu^2}$, and again t, α, β, γ, δ, are uniform functions of our new variable η, functions, it may be added, which exist only in the upper half of the η-plane. *Hence η is the uniformizing variable which we have been seeking, the variable which plays the rôle taken by t in our discussion of the special problem.*

We turn therefore to the consideration of t, α, β, γ, δ, regarded as functions of η.

The variable t is affected *additively* by the linear substitutions of η which correspond to the successive reproductions of the figure; *i.e.* with every substitution it is increased by a constant. More-

over, it becomes infinite, and that simply infinite algebraically, at all those points of the η-plane which correspond to the point e_∞ of the u-plane, the points, namely, which are equivalent to the single angular point marked e_∞ in the hexagon of our figure.

On the other hand, α, β, γ, δ, are affected multiplicatively by the linear substitutions of η. Each becomes zero in one series of equivalent points, and that simply, and each becomes infinite, and that also simply, in another series of equivalent points. The ∞-points are the same as those for which t becomes infinite; the 0-points are the points on the perimeters of our hexagons which correspond to the four points $u = \pm 1$ of our original Riemann surface of two sheets on the u-plane. The two points corresponding to $u = +1$ we may name a', a'', and the two points corresponding to $u = -1$, b', b'', in such a manner that the series of equivalent 0-points of α, β, γ, δ, correspond respectively to a', b', b'', a''.

On this characterization of our functions $t, \alpha, \beta, \gamma, \delta$, we have now to base their analytical representation in terms of η. This is to be accomplished by means of the functions which in this more general case of the automorphic theory play the same fundamental rôle as the elliptic σ-functions in the more elementary case — the so-called *prime-forms*. The *prime-*

form is not a function of η, *but a homogeneous function of the first degree of* η_1, η_2 (*where* $\frac{\eta_1}{\eta_2} = \eta$); *like the elliptic* σ-*function, it vanishes at all of a certain series of equivalent points, and is nowhere infinite.*

I use the name prime-form because all the algebraic integral forms belonging to the Riemann surface admit of being similarly expressed as products of suitably chosen prime-forms, just as in ordinary arithmetic integers as products of prime numbers. It may be added that these prime-forms are not completely determined quantities. They may be altered by certain factors, the exact expression of which here would cause too serious a digression.

If now we represent the prime-form whose zero-points are the series of equivalent points corresponding to the point m of the Riemann surface by the symbol $\Sigma(\eta_1, \eta_2; m)$, we have the following analytical representation of our functions t, α, β, γ, δ, viz.:

$$t = \frac{\Sigma'(\eta_1, \eta_2;\ e_\infty)}{\Sigma(\eta_1, \eta_2;\ e_\infty)},$$

$$\alpha = \frac{\Sigma(\eta_1, \eta_2;\ a')}{\Sigma(\eta_1, \eta_2;\ e_\infty)}, \qquad \beta = \frac{\Sigma(\eta_1, \eta_2;\ b')}{\Sigma(\eta_1, \eta_2;\ e_\infty)},$$

$$\gamma = \frac{\Sigma(\eta_1, \eta_2;\ b'')}{\Sigma(\eta_1, \eta_2;\ e_\infty)}, \qquad \delta = \frac{\Sigma(\eta_1, \eta_2;\ a'')}{\Sigma(\eta_1, \eta_2;\ e_\infty)}.$$

And so we find here, as before, that *the functions* α, β, γ, δ *prove to be the simplest elements for the*

representation of the top's motion. They are the simplest quotients of the elementary functions of the "hyperelliptic body" which has replaced the "elliptic body" of our earlier discussion.

It may be remarked that these formulas at once reduce to $t = \eta$ and the previously obtained elliptic formulas on making the hypotheses $P \gtrless 0$, $s = 0$, which are equivalent to supposing the point of support fixed.

Moreover, it must be said that these expressions for t, α, β, γ, δ are only to be understood as having a formal significance. There is altogether lacking the actual determination of the constants left at our disposal by the definitions of the Σ's, and which, it may be added, differs for the different Σ's which appear in the formulas for t, α, β, γ, δ.

And with this we come upon the point at which this theory is still incomplete. The exact determination of the formulas, and in general the means of reckoning them out by practicable methods, are for the most part wanting. The theory of the automorphic functions which for a time was a matter of principal interest in the theory of functions has in recent years not attracted the attention nor found the support which it seems to deserve. *I have therefore the more gladly laid stress here on the fact that these are not only functions possessing a theoretical*

interest, but functions which necessarily present themselves if one will completely solve even the simplest problems of mechanics.

Had we the time, we should find it interesting to consider the geometry of this more general case of the top's motion also.

I will, however, give the equation of the curve traced on the horizontal plane by the point of support. It is $x + iy = 2\alpha\beta s$, as results from the formulas on page 8, by giving X, Y, Z the values of 0, 0, $-s$, respectively. For the values of x and y depend on α, β, γ, δ alone, these quantities, in the present case, conditioning the motion of the centre of gravity up and down its vertical, and no terms appearing in the expressions for x and y due to this motion.

And I may also make the general remark that in this geometrical study the non-Euclidean interpretation plays an important rôle. For while the curves traced by the apex, etc., have in real time a form quite similar to that in the case of the fixed point of support, the Riemann surface as described by the apex on the fixed sphere brings fully into evidence the difference between the elliptic and hyperelliptic characters of the two motions. Instead of the quadrilateral which was represented in Fig. 8 we should here have a hexagon.